

# SOME EXPLICIT FORMULAE FOR THE SINGULAR VALUES OF CERTAIN HANKEL OPERATORS WITH FACTORIZABLE SYMBOL\*

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**Abstract.** In this paper a determinantal formula is written that allows one to compute the singular values of Hankel operators, the  $L^\infty$ -symbols of which are of the form  $\bar{m}w$  for  $w \in H^\infty$  rational and  $m \in H^\infty$  inner. (All of the Hardy spaces are defined on the unit circle in the usual way.) This is related, moreover, to some problems from control and systems theory.

**Key words.** Hankel operator, compressed shift, discrete spectrum, singular values,  $H^\infty$ -optimization

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**1. Introduction.** In the past few years there has been a substantial literature devoted to the computation of the norm and, more generally, singular values of Hankel operators, the  $L^\infty$ -symbol of which is of the form  $\bar{m}w$  for  $w \in H^\infty$  rational and  $m \in H^\infty$  inner. (All of our Hardy spaces will be defined on the open unit disc  $D$  following the standard conventions of [9].) A partial reference list of this work can be found in the monograph [5].

A strong motivation for studying this problem comes from control engineering, e.g., from  $H^\infty$ -optimal sensitivity theory, and from Hankel norm approximation problems in system design. (Once again we refer the interested reader to [5] for the relevant physical background.)

This paper is based on the authors' previous work [2]–[4] and [10]. We put these ideas together here, and write an elementary procedure for the computation of the singular values of the above operators based on a determinantal formula, which we derive in § 3 (see (3.8)).

More precisely, let us take our point of view from [8] and [9]. Given  $m \in H^\infty$  inner and nonconstant let  $H^2 \ominus mH^2$  denote the orthogonal complement of  $mH^2$  in  $H^2$ , and let  $P: H^2 \rightarrow H^2 \ominus mH^2 =: H$  denote the orthogonal projection. Given  $w \in H^\infty$ ,  $M_w: H^2 \rightarrow H^2$  denotes the operator induced by multiplication by  $w$ . We now set  $w(T) := PM_w|H$ . In particular,  $T := PS|H$  for  $S: H^2 \rightarrow H^2$ , the unilateral right shift. ( $T$  is called the *compressed shift*.) Then it is completely standard to show [7] that in order to solve the aforementioned Hankel singular value problem, we can equivalently find the singular values of  $w(T)$ .

In point of fact, the more general problem we solve in this paper is the rather explicit computation of the discrete spectrum of operators of the form  $w(T)w(T)^*$ , where  $T \in C_0(1)$  and  $w \in H^\infty$  is rational. Recall from [9] that a contraction  $T$  on a Hilbert space  $H$  is of class  $C_0(1)$  if  $T^n \rightarrow 0$  and  $T^{*n} \rightarrow 0$  strongly, and the operators (the squares of the "defect" operators)  $I - TT^*$  and  $I - T^*T$  have rank 1. Such operators appear in great abundance in mathematics and in a number of physical problems. See the recent treatise [6].

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Although the methods we employ in this note are basically operator theoretic, there is an important algebraic constituent as well, which allows us via a determinantal formula (see (3.8) below) to explicitly determine certain invertible elements of the noncommutative ring of operators  $\mathbf{C}[T, T^*]$ . We believe this is the main mathematical contribution of this paper.

Hopefully, some other uses will be found for our methods, both from the theoretical and applied points of view. In particular, we believe it would be very interesting to digitally implement some of the formulae given in § 3.

**2. Problem statement and preliminary results.** As we noted in the Introduction, we are interested in determining the discrete spectrum of  $w(T)w(T)^*$  for  $w \in H^\infty$  rational, and  $T$  the compressed shift associated to the nonconstant inner function  $m \in H^\infty$ . Recall that the discrete spectrum of a bounded self-adjoint operator  $B$ , denoted by  $\sigma_d(B)$ , consists of the isolated points of the spectrum  $\sigma(B)$ , which are eigenvalues of finite multiplicity. For such an operator  $B$ , the essential spectrum (denoted by  $\sigma_e(B)$ ) is the complement of  $\sigma_d(B)$  in the spectrum  $\sigma(B)$ . (See, e.g., [6] for a more detailed discussion.) We should also mention that for the compressed shift  $T$ , we have that  $\sigma_d(T) = \sigma(T) \cap D$  (the eigenvalues of finite multiplicity), while for the essential spectrum we have  $\sigma_e(T) = \sigma(T) \cap \partial D$ , where  $D$  denotes the open unit disc and  $\partial D$  the unit circle (see [6], [9]).

Now in order to avoid some (minor) technical difficulties we will assume throughout this paper that  $w$  is *not* a constant multiple of a Blaschke product. Indeed, in the event  $w$  is a constant times a (finite) Blaschke product, all of the “ $s$ -numbers” of the Hankel associated to  $\bar{m}w$  will be equal to  $\|w\|_\infty$  when  $\deg m > \deg w$  (see [1], [6]). Thus the interesting case of irrational  $m$  is easily solved. Moreover, the case of  $\deg m \leq \deg w$  can be handled using classical Nevanlinna–Pick interpolation theory.

We now express  $w = p/q$  as a ratio of relatively prime polynomials, and we set  $n := \max \{\deg p, \deg q\}$ . For  $\rho \in \mathbf{R}$ , let

$$P_\rho := q(T) \left( I - \frac{1}{\rho^2} w(T)w(T)^* \right) q(T)^* = q(T)q(T)^* - \frac{1}{\rho^2} p(T)p(T)^*.$$

We can clearly write

$$P_\rho = \sum_{k,j=0}^n C_{kj}^\rho T^k T^{*j}$$

for some constants  $C_{kj}^\rho$  with the property

$$(1) \quad C_{kj}^\rho = \bar{C}_{jk}^\rho.$$

For  $z \in \mathbf{C}$ , define

$$(2) \quad \phi_\rho(z, \bar{z}) := \sum_{k,j=0}^n C_{kj}^\rho z^k \bar{z}^j$$

and note that

$$\phi_\rho(T, T^*) = P_\rho.$$

Next  $\rho^2 \in \sigma(w(T)w(T)^*)$  if and only if  $0 \in \sigma(\phi_\rho(T, T^*))$ . Further  $\rho^2 \in \sigma_d(w(T)w(T)^*)$  if and only if  $0 \in \sigma_d(\phi_\rho(T, T^*))$ , and similarly for  $\sigma_e$ .

We now will prove some preliminary lemmas that we will need in order to make some reductions in our computation of  $\sigma_d(w(T)w(T)^*)$ . Our first result is Lemma 2.1.

LEMMA 2.1.  $T = V + F$  where  $V$  is unitary, and the rank of  $F$  is finite. Moreover  $\sigma_e(V) = \sigma_e(T) = \sigma(T) \cap \partial D$ .

*Proof.* Set

$$V := T \left( I - \frac{1}{\|\mu\|^2} \mu \otimes \mu \right) + \frac{1}{\|\mu\|^2} \mu_* \otimes \mu$$

where

$$(a \otimes b)c := \langle c, b \rangle a$$

for  $a, b, c \in H := H^2 \ominus mH^2$ ,  $\mu(\zeta) := \bar{\zeta}(m(\zeta) - m(0))$ , and  $\mu_*(\zeta) := 1 - m(\zeta)\overline{m(0)}$ . Then it is easy to check that  $V$  is unitary,  $\text{rank } F \leq 2$ , and  $\sigma_e(V) = \sigma_e(T) = \sigma(T) \cap \partial D$  (since  $V$  and  $T$  differ by a finite rank perturbation).  $\square$

Since we have assumed that  $w$  is not a constant times a Blaschke product, we see that

$$(3) \quad \phi_\rho|_{\partial D} \neq 0.$$

Set  $\hat{\phi}_{0\rho}(\zeta) := \phi_\rho(\zeta, \bar{\zeta})$  for  $\zeta \in \partial D$  (the unit circle). Then we have Corollary 2.2.

COROLLARY 2.2.  $0 \in \sigma_e(P_\rho)$  if and only if  $\{\zeta \in \partial D: \hat{\phi}_{0\rho}(\zeta) = 0\} \cap \sigma(T) \neq \emptyset$ .

*Proof.* From (2.1) we get that  $P_\rho = \phi_\rho(T, T^*) = \hat{\phi}_{0\rho}(V) + Q$ , where  $Q$  is a finite rank operator. Thus from the fact that  $V$  is unitary and our above discussion, we see

$$\sigma_e(P_\rho) = \sigma_e(\hat{\phi}_{0\rho}(V)) = \hat{\phi}_{0\rho}(\sigma_e(V)) = \{\hat{\phi}_{0\rho}(\zeta): \zeta \in \sigma(T) \cap \partial D\},$$

which immediately implies our result.  $\square$

Remark 2.3. (i) Corollary 2.2 implies that in order to determine if  $\rho^2 \in \sigma_d(w(T)w(T)^*)$  we can always assume that

$$(4) \quad \{\zeta \in \partial D: \hat{\phi}_{0\rho}(\zeta) = 0\} \cap \sigma(T) = \emptyset.$$

(ii) Let  $\rho_{\text{ess}} := \rho_{\text{ess}}(w(T))$  denote the essential norm of  $w(T)$  (i.e., the distance of  $w(T)$  to the space of compact operators on  $H$ ). Then we can show that [6], [9]

$$\begin{aligned} \rho_{\text{ess}}^2 &= \sup \{ |w(\lambda)|^2: \lambda \in \sigma_e(T) \} \\ &= \sup \{ |w(\lambda)|^2: \lambda \text{ is a singular point of } m \text{ on } \partial D \}. \end{aligned}$$

Notice that if  $\rho > \rho_{\text{ess}}$ , then automatically the assumption (4) given in (i) is satisfied. Moreover, the points in the set  $\sigma(w(T)w(T)^*) \cap (\rho_{\text{ess}}^2, \infty)$  are precisely squares of the singular values of the operator  $w(T)$ . These points are part of the discrete spectrum of  $w(T)w(T)^*$ .

In summary, we have shown in this section that the computation of the discrete spectrum of  $w(T)w(T)^*$  amounts to determining whether zero is an eigenvalue of finite multiplicity of the operator  $\phi_\rho(T, T^*) \in \mathcal{C}[T, T^*]$  for given  $\rho \in \mathbb{R}$ , where  $\phi_\rho$  enjoys the properties (1)–(4). This is precisely the problem that we solve in the next section.

**3. Main results.** In this section we will formulate and prove our theorem on the computation of the discrete spectrum of operators of the form  $w(T)w(T)^*$ . From our discussion in § 2 we are reduced to the following kind of operator theoretic problem.

Let

$$\phi(z, \bar{z}) = \sum_{k,j=0}^n C_{kj} z^k \bar{z}^j \quad (z \in \mathbb{C})$$

be a polynomial with the following properties:

- (i)  $\phi(z, \bar{z}) = \overline{\phi(\bar{z}, z)}$ , i.e.,  $C_{kj} = \bar{C}_{jk}$  ( $0 \leq j, k \leq n$ ).
- (ii)  $\phi|_{\partial D} \neq 0$ . Set  $\hat{\phi}_0(\zeta) := \phi(\zeta, \bar{\zeta})$ ,  $\zeta \in \partial D$ .
- (iii)  $\{\zeta \in \partial D: \hat{\phi}_0(\zeta) = 0\} \cap \sigma(T) = \emptyset$ .



Now set

$$A := \phi(T, T^*) = \sum_{k,j=0}^n C_{kj} T^k T^{*j}.$$

From our arguments in § 2, we need to find a computable procedure for determining whether  $0 \in \sigma_d(A)$ . This will be done via a determinantal formula given in (3.6) and (3.8). For convenience, we now state the following reformulation of (2.2).

LEMMA 3.1.  $0 \notin \sigma_e(A)$ . Equivalently  $0 \in \sigma_d(A)$  if and only if  $0 \in \sigma(A)$ .

*Proof.* This follows immediately from (2.2) and property (iii) above.  $\square$

Now in order to give our determinantal formula we will first have to compute the action of  $A$  on an element  $g \in H := H^2 \ominus mH^2$ . Accordingly, let

$$g = g_0 + g_1 \zeta + \cdots \quad (\zeta \in \partial D),$$

$$\bar{m}g = g_{-1} \bar{\zeta} + g_{-2} \bar{\zeta}^2 + \cdots.$$

Then

$$T^j g = P(\zeta^j g) = \zeta^j g - m(\zeta^{j-1} g_{-1} + \cdots + g_{-j}),$$

$$T^{*j} g = \bar{\zeta}^j g - (\bar{\zeta}^j g_0 + \cdots + \bar{\zeta} g_{j-1})$$

for  $j \geq 1$ , and where  $P: L^2 \rightarrow H$  denotes orthogonal projection.

Consequently,

$$\begin{aligned} Ag &= \sum_{k,j=0}^n C_{kj} P(\zeta^k T^{*j} g) \\ &= \sum_{k,j=0}^n C_{kj} P(\zeta^{k-j} g) - \sum_{j>0} C_{kj} \sum_{l=0}^{j-1} g_l P \zeta^{k-j+l} \\ &= \hat{\phi}_0(\zeta) g - \sum_{k>j} C_{kj} m \sum_{l=1}^{k-j} g_{-l} \zeta^{k-j-l} \\ &\quad - \sum_{k<j} C_{kj} \sum_{l=0}^{j-k-1} g_l \bar{\zeta}^{j-k-l} - \sum_{\substack{j>0 \\ k-j+l>0 \\ j>l}} C_{kj} g_l P \zeta^{k-j+l} \\ &= \hat{\phi}_0(\zeta) g - \sum_{l=1}^n g_{-l} m \sum_{k-j \geq l} C_{kj} \zeta^{k-j-l} \\ &\quad - \sum_{l=0}^{n-1} g_l \sum_{j-k > l} C_{kj} \bar{\zeta}^{j-k-l} - \sum_{l=0}^{n-1} g_l \sum_{k+l \geq j > l} C_{kj} P \zeta^{k+l-j}. \end{aligned}$$

Set

$$(5) \quad \phi_l^+(\zeta) := \sum_{k-j \geq l} C_{kj} \zeta^{k-j-l} \quad (1 \leq l \leq n),$$

$$(6) \quad \phi_l^-(\zeta) := \sum_{j-k > l} C_{kj} \bar{\zeta}^{j-k-l} \quad (0 \leq l \leq n-1),$$

$$\begin{aligned} (7) \quad \phi_l(\zeta) &:= \sum_{k+l \geq j > l} C_{kj} P \zeta^{k+l-j} \quad (0 \leq l \leq n-1) \\ &= \sum_{k+l \geq j > l} C_{kj} (P(\zeta^{k+l-j} P 1))(\zeta) \\ &= \sum_{k+l \geq j > l} C_{kj} [\zeta^{k+l-j} \mu_*(\zeta) - m(\zeta^{k+l-j-1} \mu_{*, -1} + \cdots + \mu_{*, -(k+l+j)})] \end{aligned}$$

where

$$(8) \quad \mu_{*, -j} := \bar{m}_j \quad \text{for } j \geq 1.$$

(We are setting  $m(\zeta) = \sum_{j=0}^{\infty} m_j \zeta^j$ .)

We can now summarize our above computation by Lemma 3.2.

LEMMA 3.2. *We have for  $g \in H$  that*

$$(9) \quad Ag = \hat{\phi}_0 g - \sum_{l=1}^n g_{-l} m \phi_l^+ - \sum_{l=0}^{n-1} g_l \phi_l^- - \sum_{l=0}^{n-1} g_l \phi_l$$

where  $\hat{\phi}_0$ ,  $\phi_l^-$ ,  $\phi_l^+$ , and  $\phi_l$  are explicitly given from (ii), (5), (6), (7), respectively, above.

Remark 3.3. Notice from (3.1) that  $0 \in \sigma(A)$  if and only if there exists  $g \in H$ ,  $g \neq 0$  such that  $Ag = 0$ . From (3.2) we can compute the action of  $A$  on  $g$ . We assume from now on that  $Ag = 0$ . Then by (3.2) (for  $Ag = 0$ ), we have that

$$(10) \quad \hat{\phi}_0 g = \sum_{l=1}^n g_{-l} m \phi_l^+ + \sum_{l=0}^{n-1} g_l (\phi_l^- + \phi_l).$$

Now define

$$(11) \quad \psi(z) := \sum_{k,j=0}^n C_{kj} z^{n+k-j} \quad (z \in \mathbb{C}).$$

Then multiplying (10) by  $\zeta^n$ , we can easily deduce that

$$(12) \quad \psi(z)g(z) = \sum_{l=1}^n g_{-l} m(z) \phi_l^+(z) z^n + \sum_{l=0}^{n-1} g_l (\psi_l^-(z) + z^n \phi_l(z))$$

for  $z = \zeta \in \partial D$ , where

$$(13) \quad \psi_l^-(z) := \sum_{j-k > l} C_{kj} z^{n+l+k-j}$$

for  $z \in \mathbb{C}$ .

We now make a technical assumption in order to simplify our exposition. This assumption of genericity will be removed when we state our final result in (3.8).

Assumption 3.4. All the zeros of  $\psi$  are distinct and different from zero.

We now come to the following result.

LEMMA 3.5. *Under assumption (3.4), there exist  $z_1, z_2, \dots, z_p \in D$ ,  $\zeta_1, \zeta_2, \dots, \zeta_q \in \partial D \setminus \sigma(T)$ ,  $2p+q=2n$ , such that  $\psi(z) = \alpha(z-z_1) \cdots (z-z_p)(z-1/\bar{z}_1) \cdots (z-1/\bar{z}_p)(z-\zeta_1) \cdots (z-\zeta_q)$  for some  $\alpha \neq 0$ .*

Proof. From properties (i) and (ii) at the beginning of this section we have that

$$\begin{aligned} z^{2n} \overline{\psi(1/\bar{z})} &= z^{2n} \sum_{k,j=0}^n \overline{C_{kj}} (1/\bar{z})^{n+k-j} \\ &= \sum_{k,j=0}^n \bar{C}_{kj} z^{n-k+j} \\ &= \sum_{k,j=0}^n C_{jk} z^{n-k+j} = \psi(z). \end{aligned}$$

From (ii),  $\psi \neq 0$ . Denote by  $\zeta_1, \zeta_2, \dots, \zeta_q$  the zeros of  $\psi$  on  $\partial D$ . From our above computation it follows that if  $\psi(z_0) = 0$  for  $z_0 \in D$  with  $z_0 \neq 0$ , then  $\psi(1/\bar{z}_0) = 0$  also. This yields the representation of  $\psi$ . Finally  $\zeta_j \notin \sigma(T)$  by (iii) for  $j = 1, \dots, q$ .  $\square$

We are almost done! Indeed all the functions in (12) are analytic in a neighborhood of  $D$  except  $\sigma(T) \cap \partial D$ . This allows us to set  $z = z_1, z_2, \dots, z_p, \zeta_1, \dots, \zeta_q$  in (12) obtaining

$$(14) \quad \sum_{l=1}^n g_{-l} m(z_r) \phi_l^+(z_r) z_r^n + \sum_{l=0}^{n-1} g_l (\psi_l^-(z_r) + z_r^n \phi_l(z_r)) = 0 \quad \text{for } 1 \leq r \leq p,$$

$$(15) \quad \sum_{l=1}^n g_{-l} m(\zeta_s) \phi_l^+(\zeta_s) \zeta_s^n + \sum_{l=0}^{n-1} g_l (\psi_l^-(\zeta_s) + \zeta_s^n \phi_l(\zeta_s)) = 0 \quad \text{for } 1 \leq s \leq q.$$

Now multiplying (10) by  $\overline{\zeta^n m(\zeta)}$ , we see

$$(16) \quad [\bar{z}^n \hat{\phi}_0](z) (\bar{m}g)(z) = \sum_{l=1}^n g_{-l} [\bar{z}^n \phi_l^+](z) + \sum_{l=0}^{n-1} g_l \{ [\bar{z}^n \bar{m} \phi_l^-](z) + [\bar{z}^n \bar{m} \phi_l](z) \}$$

where  $z = \zeta \in \partial D$  and all the functions are analytic in  $\bar{z}$ . Note that even though this equation has been derived on  $\partial D$ , it is valid on the complement of  $D$  if we replace  $\bar{z}$  by  $1/z$  for  $|z| > 1$ .

Now set

$$(17) \quad \psi_l^+(z) := (\bar{z}^n \phi_l^+)(z) = \sum_{k-j \geq l} C_{kj} \bar{z}^{n+j-k+l} \quad (1 \leq l \leq n)$$

and

$$(18) \quad \begin{aligned} \psi_l(z) &:= \bar{z}^n (\bar{m} \phi_l)(z) \quad (0 \leq l \leq n-1) \\ &= \sum_{k+l \geq j > l} C_{kj} \bar{z}^{n+j-k-l} [(\overline{m(z)} - \overline{m(0)}) - (\bar{z}^{n-k-l+j+1} \mu_{*, -1} \\ &\quad + \dots + \bar{z}^n \mu_{*, -(k+l+j)})] \end{aligned}$$

for  $z = \zeta \in \partial D$ . Once again  $\psi_l^+(z)$ ,  $\psi_l(z)$  admit analytic extensions to the complement of  $\bar{D}$  if we replace  $\bar{z}$  by  $1/z$  for  $|z| > 1$ .

Moreover for  $1 \leq r \leq p$ ,

$$(19) \quad \bar{z}_r^n \hat{\phi}_0 \left( \frac{1}{\bar{z}_r} \right) = \bar{z}_r^{2n} \psi \left( \frac{1}{\bar{z}_r} \right) = 0.$$

(Notice we are setting  $\hat{\phi}_0(z) := \phi(z, 1/z)$  for  $z \in \mathbb{C}$ .) Then from (16) we see that for  $1 \leq r \leq p$ ,

$$(20) \quad \sum_{l=1}^n g_{-l} \psi_l^+ \left( \frac{1}{\bar{z}_r} \right) + \sum_{l=0}^{n-1} g_l \left\{ \bar{z}_r^{-n} \bar{m} \phi_l \left( \frac{1}{\bar{z}_r} \right) + \psi_l \left( \frac{1}{\bar{z}_r} \right) \right\} = 0.$$

(We are setting  $\bar{m}(z) := \overline{m(1/\bar{z})}$  for  $|z| > 1$ .)

Finally we note that if  $g_{-n} = \dots = g_{n-1} = 0$ , then from (3.2) we have that  $\phi_0 g = 0$  (note we have taken  $g$  such that  $Ag = 0$ ), which by property (ii) above implies that  $g = 0$ .

We now come to the final point in our computations. Namely the above argument shows that  $0 \in \sigma(A)$  if and only if the characteristic determinant of the  $2n$  equations in the  $2n$  "unknowns"  $g_{-n}, \dots, g_{n-1}$  is zero. We can write this determinant quite explicitly. Indeed in order to do this, let us introduce the notation

$$(21) \quad M(\theta_1, \dots, \theta_R; \xi_1, \dots, \xi_N) = \begin{bmatrix} \theta_1(\xi_1) & \dots & \theta_R(\xi_1) \\ \vdots & & \vdots \\ \theta_1(\xi_N) & \dots & \theta_R(\xi_N) \end{bmatrix}$$

for functions  $\theta_1, \dots, \theta_R$  well defined in a neighborhood of  $\xi_1, \dots, \xi_N$  with the  $\xi_i$  distinct for  $i = 1, \dots, N$ .

Using this notation, our preceding arguments prove the following theorem.

**THEOREM 3.6.** *Under Assumption 3.4,  $0 \in \sigma(A)$  if and only if*

$$(22) \quad \det \begin{bmatrix} M^- & M^+ \\ N^- & N^+ \\ M_*^- & M_*^+ \end{bmatrix} = 0$$

where

$$(23) \quad M^- := M(z^n m \phi_n^+, \dots, z^n m \phi_1^+; z_1, \dots, z_p),$$

$$(24) \quad M^+ := M(\psi_0^- + z^n \phi_0, \dots, \psi_{n-1}^- + z^n \phi_{n-1}; z_1, \dots, z_p),$$

$$(25) \quad M_*^- := M(\psi_n^+, \dots, \psi_1^+; 1/\bar{z}_1, \dots, 1/\bar{z}_p),$$

$$(26) \quad M_*^+ := M(\psi_0 + z^{-n} \bar{m} \phi_0^-, \dots, \psi_{n-1} + z^{-n} \bar{m} \phi_{n-1}^-; 1/\bar{z}_1, \dots, 1/\bar{z}_p).$$

$N^-$  and  $N^+$  are defined as in (23) and (24) by replacing  $z_1, \dots, z_p$  with  $\xi_1, \dots, \xi_q$ .

*Proof.* Write the characteristic determinant of the system of equations in (14), (15), (20)!  $\square$

**Remark 3.7.** We will now eliminate Assumption 3.4 in Theorem 3.6. Note that if the roots of  $\psi$  are not distinct perturbing  $\psi$  by  $\varepsilon$ ,  $\varepsilon$  a suitable sufficiently small number, will assure that the corresponding  $\psi_\varepsilon$  does have distinct roots different from zero.

Before stating our result, we will need to extend the definition of  $M$  in (21) to the case where the  $\xi_i$  have multiplicities. Indeed, we set

$$(27) \quad M(\theta_1, \dots, \theta_R; \xi_1, \dots, \xi_1, \dots, \xi_S, \dots, \xi_S) := \begin{bmatrix} \theta_1(\xi_1) & \theta_2(\xi_1) & \dots & \theta_R(\xi_1) \\ \theta'_1(\xi_1) & \theta'_2(\xi_1) & \dots & \theta'_R(\xi_1) \\ \vdots & \vdots & \ddots & \vdots \\ \theta_1^{(N_1-1)}(\xi_1) & \theta_2^{(N_1-1)}(\xi_1) & \dots & \theta_R^{(N_1-1)}(\xi_1) \\ \dots & \dots & \dots & \dots \\ \theta_1(\xi_S) & \theta_2(\xi_S) & \dots & \theta_R(\xi_S) \\ \vdots & \vdots & \ddots & \vdots \\ \theta_1^{(N_S-1)}(\xi_S) & \theta_2^{(N_S-1)}(\xi_S) & \dots & \theta_R^{(N_S-1)}(\xi_S) \end{bmatrix}$$

where  $\xi_i$  has multiplicity  $N_i$  for  $i = 1, \dots, S$  in  $M$ , and the functions  $\theta_1, \dots, \theta_R$  are analytic in a neighborhood of  $\xi_1, \dots, \xi_S$ . (For  $\theta$  analytic,  $\theta^{(N)}$  denotes the derivative of order  $N$ .)

Then taking  $\varepsilon \rightarrow 0$  in our above argument, we easily get the following corollary.

**COROLLARY 3.8.** *In complete generality (i.e., without Assumption 3.4), we have that  $0 \in \sigma(A)$  if and only if the determinant (22) is zero where we use the definition (27) of  $M$  in (23)–(26) above, and each root of  $\psi(z)$  is counted according to its multiplicity.*

**Remark 3.9.** (i) The determinantal formula (22) gives us an explicit expression for determining the invertibility of the operator  $A \in \mathcal{C}[T, T^*]$ . Moreover from our discussion in § 2, we can now also find  $\sigma_d(w(T)w(T)^*)$ . We will apply this to an example in § 4.

(ii) We should also note that the multiplicity of zero as a root of  $\det M$  (in (22)) is its multiplicity as an eigenvalue of  $A$ . Moreover, from (10) we have a formula for the corresponding *eigenvectors*. Hence we have a general procedure for computing the multiplicity of the singular values of  $w(T)$  and for the corresponding Schmidt vectors.



**4. Example.** In this section we illustrate via an example some of the computational issues involved in the determinantal scheme for computing the singular values of the operators  $w(T)$  that we have discussed above. We are convinced that it will be possible in the near future to implement on a computer the formulae discussed in § 3, so that hopefully these ideas can become of practical use for some applied problems.

Let  $w(z) = z^2 + 1$ , and let  $m \in H^\infty$  be a nonconstant inner function. We want to study the singular values of the corresponding operator  $w(T)$ . First note that  $\|w\|_\infty = 2$ , and  $w$  attains its maximum at  $\pm 1$ . If  $\pm 1 \in \sigma_e(T)$  (i.e., if  $m$  is singular at  $\pm 1$ ), then all of the  $s$ -numbers of  $w(T)$  will be equal to two. Consequently, we will assume for now on that  $\pm 1 \notin \sigma_e(T)$ , and  $\rho_{\text{ess}} := \rho_{\text{ess}}(w(T)) < 2$ . We will study the invertibility of  $P_\rho$  (notation as § 2) for  $\rho$  contained in the interval  $(\rho_{\text{ess}}, 2)$ .

The computation of the determinantal formula for the singular values of  $w(T)$  in  $(\rho_{\text{ess}}, 2)$  is now quite elementary following the arguments of § 3. Indeed using the notation of § 2 (see (11)), we have that

$$\psi_\rho(z) = -\frac{1}{\rho^2} z^4 - \left(\frac{2}{\rho^2} - 1\right) z^2 - \frac{1}{\rho^2}.$$

We can calculate that in the interval of interest all of the roots of  $\psi_\rho(z)$  lie on  $\partial D$  and are distinct. The exact formulae for these roots are  $\zeta_1 = e^{i\theta/2}$ ,  $\zeta_2 = -e^{i\theta/2}$ ,  $\zeta_3 = e^{-i\theta/2}$ ,  $\zeta_4 = -e^{-i\theta/2}$ , where

$$\theta := \arctan \frac{\rho \sqrt{1 - \rho^2/4}}{(1 - \rho^2/2)} \quad (0 < \theta < \pi/2).$$

(Notice that  $\zeta_3 = \bar{\zeta}_1$ ,  $\zeta_4 = \bar{\zeta}_2$ .)

If we now follow the recipe of § 3, we see (the computations actually were quite easy!) that the singular values of  $w(T)$  in the interval  $(\rho_{\text{ess}}, 2)$  may be derived from the determinant of the following  $4 \times 4$  matrix:

$$(28) \quad M = [N^- \quad N^+]$$

where

$$N^- := \begin{bmatrix} -(1/\rho^2) \zeta_1^2 m(\zeta_1) & -(1/\rho^2) \zeta_1^3 m(\zeta_1) \\ \vdots & \vdots \\ -(1/\rho^2) \zeta_4^2 m(\zeta_4) & -(1/\rho^2) \zeta_4^3 m(\zeta_4) \end{bmatrix},$$

$$N^+ = \begin{bmatrix} -(1/\rho^2)(1 + \zeta_1^2 \mu_*(\zeta_1)) & -(1/\rho^2)(\zeta_1 + \zeta_1^2(\zeta_1 \mu_*(\zeta_1) - m(\zeta_1) \bar{m}_1)) \\ \vdots & \vdots \\ -(1/\rho^2)(1 + \zeta_4^2 \mu_*(\zeta_4)) & -(1/\rho^2)(\zeta_4 + \zeta_4^2(\zeta_4 \mu_*(\zeta_4) - m(\zeta_4) \bar{m}_1)) \end{bmatrix}.$$

(Recall that  $\mu_*(\zeta) = 1 - m(\zeta) \overline{m(0)}$ , and  $m_1 := dm/d\zeta|_{\zeta=0}$ .)

Notice that in this formula  $m$  appears as a "parameter," and that, in general, the size and complexity of the matrix given in (22) above only depends on the "weighting" function  $w$ . We checked (28) for the trivial case of  $m(z) = z$ , and got (of course) the obvious answer that the unique root of  $\det M$  in  $(0, 2)$  is 1.

In conclusion, the determinantal formula (22) offers a very general theoretical procedure for the computation of the discrete spectrum of operators of the form  $w(T)w(T)^*$  for both rational and irrational inner functions  $m \in H^\infty$ . The writing of appropriate software for actually carrying this out should make a very interesting project.



## REFERENCES

- [1] V. M. ADAMJAN, D. Z. AROV, AND M. G. KREIN, *Analytic properties of Schmidt pairs for a Hankel operator and the generalized Schur-Takagi problem*, Math. USSR-Sb., 15 (1971), pp. 31-73.
- [2] C. FOIAS AND A. TANNENBAUM, *On the Nehari problem for a certain class of  $L^\infty$ -functions appearing in control theory*, J. Funct. Anal., 74 (1987), pp. 146-159.
- [3] C. FOIAS, A. TANNENBAUM, AND G. ZAMES, *On the  $H^\infty$ -optimal sensitivity problem for systems with delays*, SIAM J. Control Optim., 25 (1987), pp. 686-706.
- [4] ———, *Sensitivity minimization for arbitrary SISO distributed plants*, Systems Control Lett., 8 (1987), pp. 189-195.
- [5] B. A. FRANCIS, *A Course in  $H^\infty$  Control Theory*, Lecture Notes in Control and Information Sciences 88, Springer-Verlag, New York, 1987.
- [6] N. K. NIKOLSKII, *A Treatise on the Shift Operator*, Springer-Verlag, New York, 1986.
- [7] S. C. POWER, *Hankel Operators on Hilbert Space*, Pitman, Boston, MA, 1982.
- [8] D. SARASON, *Generalized interpolation in  $H^\infty$* , Trans. Amer. Math. Soc., 127 (1967), pp. 179-203.
- [9] B. SZ.-NAGY AND C. FOIAS, *Harmonic Analysis of Operators on Hilbert Space*, North-Holland, Amsterdam, 1970.
- [10] G. ZAMES, A. TANNENBAUM, AND C. FOIAS, *Optimal interpolation in  $H^\infty$ : a new approach*, in Proc. of the Conference on Decision and Control, December 1986, pp. 350-355.